Uniform Asymptotic Approximations for Duct Acoustic Modes in a Thin Boundary-Layer Flow

M. K. Myers* and S. L. Chuang†
The George Washington University, Hampton, Virginia

Analytical approximations for the acoustic modes in a duct carrying a uniform core flow with a thin shear layer at the walls are developed using the method of matched asymptotic expansions. Both two-dimensional and cylindrical duct propagation are considered. Numerical results for eigenvalues calculated using the theory are presented for the two-dimensional problem and compared with results from earlier analyses. It is found that the new approximations yield a significant increase in accuracy.

I. Introduction

THE problem of sound transmission through a duct carrying a uniform core flow with a thin boundary layer at the walls has been of interest for some time. Such a flow has practical applications in modeling a fully developed real flow profile. Perhaps the strongest motivation for studying the problem, however, is the possibility of obtaining analytical results using asymptotic analyses which can yield physical understanding of shear flow propagation phenomena in more realistic flows which must be treated numerically.

Analytical studies based on the assumption of small boundary-layer thickness have been carried out by several researchers during the past decade. Eversman and Beckemeyer[1] developed an inner expansion for acoustic pressure in the layer from which they were able to derive an approximate equivalent boundary condition applicable at the outer edge of the layer. Using this condition they showed that the formulation of the shear layer problem reduces to that of the associated uniform flow problem with the well-known particle displacement continuity boundary condition as the layer thickness vanishes.

In a later paper, Eversman[2] employed the equivalent boundary condition to formulate the eigenvalue problem for acoustic modes in an annular duct flow. Using the known uniform flow solutions in the core flow, an approximate eigenvalue equation is developed. Numerical solution of this single equation yields the acoustic wave numbers and circumvents the task of numerically integrating the differential equation and iterating for the shear flow eigenfunctions. The procedure is equally applicable to rectangular duct geometries, for which it leads to an eigenvalue equation equivalent to one later derived less formally by Tester.[3] A slightly different analysis by Swinbanks,[4] also for rectangular ducts, yields similar approximate results. Other studies less directly related to the present work are discussed in Sec. IV.B of Ref. 5.

The work reported in the present paper arose in the course of some research on shear flow propagation in which it was desirable to obtain analytical representations of the acoustic modes which are uniformly valid; i.e., representations which hold at a given level of accuracy over the entire width of the duct. Early in this study the equivalent boundary condition procedure of Eversman was applied to cases of two-dimensional duct propagation. In doing so it was found that the accuracy of the approximate wave numbers so obtained degrades substantially with increasing mode order unless the boundary layer is extremely thin. This is in agreement with the trends exhibited for an annular duct in Ref. 2.

An alternate asymptotic analysis of the thin shear layer eigensolutions will be given herein. It has proved to yield significantly more accurate results, especially at higher mode orders and it also leads directly to a uniform representation of the duct eigenfunctions. The results are derived by a formal application of the method of matched asymptotic expansions.[6,7] The procedure used differs somewhat from previous analyses in that it involves determination of asymptotic representations of a pair of linearly independent solutions to the equation governing the transverse acoustic modes without a priori consideration of boundary conditions. These solutions are obtained in exponential forms, which together can be interpreted as a renormalization of the series representations previously studied.[1,2] Once the fundamental set of solutions is obtained to a given order of accuracy, these solutions are subjected to boundary conditions to derive appropriate eigenvalue equations.

In Sec. II the asymptotic analysis is developed in detail for uniform two-dimensional ducts. Section III describes the parallel analysis applicable to propagation in uniform ducts of circular cross section. Then, in Sec. IV, the results are applied to derive eigenvalue equations for two particular duct configurations. Some selected numerical results are discussed in Sec. V in order to illustrate the improvement in accuracy inherent in the new theory.

II. Two-Dimensional Propagation

Figure 1 is intended to depict the flow and duct geometry in both cases of interest. The dimensionless independent variables x and y (or r) are measured in units of the duct half-width (or radius). The dimensionless boundary-layer thickness is denoted by δ, and the duct wall is located at $y=0$ (or $r=1$), where the axial mean flow velocity is assumed to vanish. For simplicity, the two-dimensional duct and flow will be assumed to be symmetric about $y=1$ with an identical boundary layer and wall at $y=2$. In the following, time will be assumed to have been made dimensionless using the speed of sound in the steady flow and the duct half-width or radius.

If the acoustic pressure is sought in the form $P(y)\exp(-i\xi x+i\omega t)$ then it is well known that the acoustic equations of motion in the presence of a parallel transversely

Presented as Paper 83-0668 at the AIAA 8th Aeroacoustics Conference, Atlanta, Ga., April 11-13, 1983; submitted April 29, 1983; revision received Nov. 7, 1983. Copyright © American Institute of Aeronautics and Astronautics, Inc., 1983. All rights reserved.

*Professor of Engineering and Applied Science, Joint Institute for Advancement of Flight Sciences. Associate Fellow AIAA.

†Research Associate, Joint Institute for Advancement of Flight Sciences; currently at Systems and Applied Sciences Corp., Hampton, Va. Member AIAA.

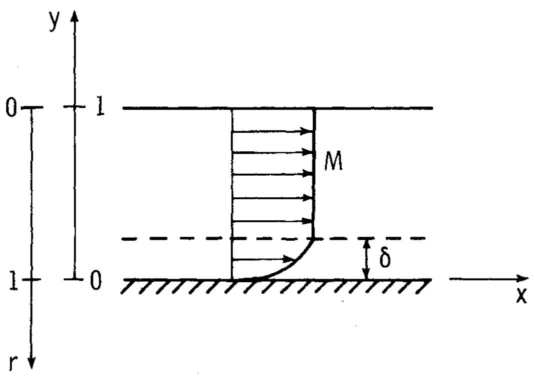

Fig. 1 Flow and duct geometry.

sheared flow along x yield for $P(y)$,

$$\frac{d^2P}{dy^2}+\frac{2\xi}{\omega-\xi M}\frac{dM}{dy}\frac{dP}{dy}+[(\omega-\xi M)^2-\xi^2]P=0 \tag{1}$$

where $M(y)$ is the flow Mach number, ξ the dimensionless complex axial wave number, and ω the dimensionless driving frequency of the sound source. The assumption is made throughout that critical layers do not exist, i.e., that $\omega-\xi M\neq 0$ for any y or ξ of interest.

The flow profile to be considered in the present work is of the form

$$M(y)=M_0\Phi(y/\delta) \tag{2}$$

in which M_0 is the constant core flow Mach number, and Φ is a specified shape function. It is assumed that

$$\Phi(y/\delta)=1, \qquad y\geq\delta$$

$$\Phi(0)=0$$

as indicated schematically in Fig. 1.

In the following, asymptotic solutions of Eq. (1) valid for $\delta\ll 1$ are to be derived using the method of matched asymptotic expansions. Thus, an inner variable $Y=y/\delta$ is introduced, and Eq. (1) is written in terms of Y. Making use of Eq. (2) this yields

$$\frac{d^2P}{dY^2}+\frac{2\xi M_0\Phi'(Y)}{\omega-\xi M_0\Phi(Y)}\frac{dP}{dY}+\delta^2[(\omega-\xi M_0\Phi(Y))^2-\xi^2]P=0 \tag{3}$$

Now, in the core region $y>\delta$, the general solution of Eq. (1) can be expressed as a linear combination of the linearly independent set of solutions

$$P=e^{\pm i\lambda y}, \qquad \lambda^2=(\omega-\xi M_0)^2-\xi^2 \tag{4}$$

The approach to be followed in solving Eq. (3) will be to determine the inner form of two linearly independent solutions, each of which asymptotically matches one of the solutions in Eq. (4). Thus, it is convenient to write

$$P=e^{\phi}$$

which, when substituted into Eq. (3), results in an equation for ϕ, considered as a function of Y, in the form

$$\phi''+(\phi')^2-(h'/h)\phi'+\delta^2(h-\xi^2)=0 \tag{5}$$

In Eq. (5) the function h has been defined to simplify writing the following analysis as

$$h(Y)=[\omega-\xi M_0\Phi(Y)]^2 \tag{6}$$

and it is noted for later purposes that, because of the assumed behavior of Φ,

$$h(Y)\equiv h_0=(\omega-\xi M_0)^2 \qquad h(Y)-\xi^2=h_0-\xi^2=\lambda^2 \tag{7}$$

for $Y\geq 1$.

To investigate the nature of solutions for P in the region $Y=\mathcal{O}(1)$ it is assumed that ϕ can be expanded in a power series in δ:

$$\phi=\sum_{n=0}^{\infty}\delta^n\phi_n(Y) \tag{8}$$

Substituting this inner expansion into Eq. (5) and equating coefficients of like powers of δ to zero results in a sequence of equations for the ϕ_n. The first four of these are

$$\mathcal{O}(\delta^0):\quad \phi_0''+(\phi_0')^2-\frac{h'}{h}\phi_0'=0$$

$$\mathcal{O}(\delta):\quad \phi_1''+2\phi_0'\phi_1'-\frac{h'}{h}\phi_1'=0$$

$$\mathcal{O}(\delta^2):\quad \phi_2''+2\phi_0'\phi_2'-\frac{h'}{h}\phi_2'=-(\phi_1')^2-(h-\xi^2)$$

$$\mathcal{O}(\delta^3):\quad \phi_3''+2\phi_0'\phi_3'-\frac{h'}{h}\phi_3'=-2\phi_1'\phi_2' \tag{9}$$

General solutions for the functions ϕ_n can be determined recursively. However, a great deal of algebra can be avoided if it is recognized at the outset that, for $\lambda\neq 0$, the inner solution $\exp(\phi)$ will match with the outer solution (4) only if the leading term ϕ_0 is independent of Y. Hence, the appropriate solution to the first of Eqs. (9) is $\phi_0'=0$ or $\phi_0=B_0$ where B_0 is a constant that represents the arbitrary amplitude of the solution $\exp(\phi)$. With $\phi_0'=0$, the solutions for the remaining ϕ_n are simply determined in the form

$$\phi_1=A_1\int_1^Y h(\eta)d\eta+B_1$$

$$\phi_2=A_2\int_1^Y h(\eta)d\eta+B_2-A_1^2\int_1^Y h(\zeta)\int_1^{\zeta}h(\eta)d\eta d\zeta-\int_1^Y h(\zeta)\int_1^{\zeta}\frac{[h(\eta)-\xi^2]}{h(\eta)}d\eta d\zeta$$

$$\phi_3=A_3\int_1^Y h(\eta)d\eta+B_3-2A_1A_2\int_1^Y h(\zeta)\int_1^{\zeta}h(\eta)d\eta d\zeta+2A_1^3\int_1^Y h(\sigma)\int_1^{\sigma}h(\zeta)\int_1^{\zeta}h(\eta)d\eta d\zeta d\sigma+2A_1\int_1^Y h(\sigma)\int_1^{\sigma}h(\zeta)\int_1^{\zeta}\frac{[h(\eta)-\xi^2]}{h(\eta)}d\eta d\zeta d\sigma \tag{10}$$

The arbitrary constants A_n and B_n in the preceding solutions are to be determined by asymptotic matching. The choice of unity for the lower limit in each integration has been made to facilitate the algebra of the matching process.

Before proceeding to the determination of the arbitrary constants, it is noted that for $Y\geq 1$ expressions (10) are considerably simplified because of relations (7). Thus, for

$Y \geq 1$,

$$\phi_1 = A_1 h_0 (Y-1) + B_1$$

$$\phi_2 = A_2 h_0 (Y-1) + B_2 - \frac{A_1^2 h_0^2}{2}(Y-1)^2 - \frac{\lambda^2}{2}(Y-1)^2$$

$$\phi_3 = A_3 h_0 (Y-1) + B_3 - A_1 A_2 h_0^2 (Y-1)^2 + \frac{A_1^3 h_0^3}{3}(Y-1)^3 + \lambda^2 \frac{A_1 h_0}{3}(Y-1)^3 \qquad (11)$$

Finally, the matching principle of Van Dyke[6] is applied to determine the constants A_n and B_n. The four-term inner expansion of ϕ is

$$\phi^{i4} = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \delta^3\phi_3$$

An outer expansion of this is constructed by expanding it for small δ keeping y fixed, i.e., by considering its form for large Y. Thus, Eqs. (11) can be used for the ϕ_n. If Y is replaced by y/δ and the result expanded for small δ up to terms of $\mathcal{O}(\delta^4)$, there results the four-term outer expansion of the four-term inner expansion, which will be denoted ϕ^{i4}_{o4}. However, because the ϕ_n of Eqs. (11) are polynomials of degree n, in the present case this process retains every term in the four-term inner expansion. Thus, when written back in terms of the inner variable Y, one finds

$$\phi^{i4}_{o4} = B_0 + \delta[A_1 h_0 (Y-1) + B_1] + \delta^2 [A_2 h_0 (Y-1) + B_2 - \tfrac{1}{2}(A_1^2 h_0^2 + \lambda^2)(Y-1)^2] + \delta^3 [A_3 h_0 (Y-1) + B_3 - A_1 A_2 h_0^2 (Y-1)^2 + (A_1 h_0/3)(A_1^2 h_0^2 + \lambda^2)(Y-1)^3] \qquad (12)$$

On the other hand, the solution (4) in the outer region implies that the outer representation of ϕ is, to any order in δ,

$$\phi^{on} = \pm i\lambda y$$

and, because of its simple form, this has an inner expansion valid to any order in δ consisting of one term only. Thus,

$$\phi^{o4}_{i4} = \pm i\lambda\delta Y$$

According to the matching principle,

$$\phi^{i4}_{o4} = \phi^{o4}_{i4}$$

Thus, equating the above expression with the result of Eq. (12), the constants A_n and B_n can be found as

$$B_0 = 0, \qquad B_1 = A_1 h_0 = \pm i\lambda$$

$$A_1 = \pm i\lambda/h_0, \qquad A_2 = A_3 = B_2 = B_3 = 0 \qquad (13)$$

Hence, by making use of Eqs. (7), (10), and (15), the inner representation for the desired pair of linearly independent solutions of Eq. (1) can be written as $P = e^{\phi}$, in which

$$\phi = \pm i\lambda\delta\left[1 + \int_1^Y \frac{h(\eta)}{h_0}\mathrm{d}\eta\right] + \delta^2\left\{\int_1^Y h(\zeta)\int_1^\zeta \left[\frac{\lambda^2}{h_0}\left(\frac{h(\eta)}{h_0} - 1\right) + \frac{\xi^2}{h_0}\left(\frac{h_0}{h(\eta)} - 1\right)\right]\mathrm{d}\eta\mathrm{d}\zeta\right\} \mp \frac{2i\lambda\delta^3}{h_0}\left\{\int_1^Y h(\sigma)\int_1^\sigma h(\zeta) \times \int_1^\zeta \left[\frac{\lambda^2}{h_0}\left(\frac{h(\eta)}{h_0} - 1\right) + \frac{\xi^2}{h_0}\left(\frac{h_0}{h(\eta)} - 1\right)\right]\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}\sigma\right\} + \mathcal{O}(\delta^4) \qquad (14)$$

Furthermore, because the inner expansion (14) contains the outer expansion to all orders, it is uniformly valid over all y and reduces exactly to

$$P = e^{\pm i\lambda y}$$

in $y > \delta$. (Note that $\phi_1 = \pm i\lambda y$, $\phi_2 = \phi_3 \equiv 0$ for $Y > 1$.)

In Sec. IV, Eq. (14) will be used to derive an approximate eigenvalue equation which determines the wave numbers ξ for a two-dimensional soft-walled duct carrying a thin boundary-layer flow. Prior to that, the corresponding analysis for a circular duct will be described in the next section.

III. Cylindrical Duct Propagation

For uniform cylindrical ducts carrying axial flow as indicated in Fig. 1, the acoustic pressure is sought in the form $P(r)\exp(-i\xi x + im\theta + i\omega t)$, leading to the equation

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \left[\frac{1}{r} + \frac{2\xi}{(\omega - \xi M)}\frac{\mathrm{d}M}{\mathrm{d}r}\right]\frac{\mathrm{d}P}{\mathrm{d}r} + \left[(\omega - \xi M)^2 - \xi^2 - \frac{m^2}{r^2}\right]P = 0 \qquad (15)$$

Here it is assumed that

$$M(r) = M_0 \Phi\left(\frac{1-r}{\delta}\right)$$

where Φ has been discussed previously.

An inner variable

$$R = (1-r)/\delta$$

is introduced, and Eq. (15) is expressed in terms of R in the form

$$\frac{\mathrm{d}^2 P}{\mathrm{d}R^2} - \left[\frac{\delta}{1-\delta R} + \frac{h'}{h}\right]\frac{\mathrm{d}P}{\mathrm{d}R} + \delta^2\left[h - \xi^2 - \frac{m^2}{(1-\delta R)^2}\right]P = 0 \qquad (16)$$

where $h(R)$ is the function defined in Eq. (6). Again solutions of Eq. (16) valid for $R = \mathcal{O}(1)$ are assumed to be

$$P = e^{\phi}, \qquad \phi = \sum_{n=0}^{\infty} \delta^n \phi_n(R) \qquad (17)$$

If the coefficients of Eq. (16) are expanded for small δ keeping R fixed, and if expressions (17) are used, then there results a sequence of equations for ϕ_n as follows:

$$\mathcal{O}(\delta^0): \quad \phi_0'' + (\phi_0')^2 - \frac{h'}{h}\phi_0' = 0$$

$$\mathcal{O}(\delta): \quad \phi_1'' + 2\phi_0'\phi_1' - \frac{h'}{h}\phi_1' = \phi_0'$$

$$\mathcal{O}(\delta^2): \quad \phi_2'' + 2\phi_0'\phi_2' - \frac{h'}{h}\phi_2' = \phi_1' - (\phi_1')^2 + R\phi_0' - (h - \xi^2 - m^2) \qquad (18)$$

where, in this case, only the three-term inner expansion of ϕ will be calculated.

The solutions of Eqs. (18) are readily determined as

$$\phi_0 = L_0$$

$$\phi_1 = K_1 \int_1^R h(\eta)\mathrm{d}\eta + L_1$$

$$\phi_2 = K_2 \int_1^R h(\eta)\mathrm{d}\eta + L_2 + K_1 \int_1^R (\eta - 1) h(\eta)\mathrm{d}\eta - K_1^2 \int_1^R h(\zeta) \int_1^\zeta h(\eta)\mathrm{d}\eta\mathrm{d}\zeta - \int_1^R h(\zeta) \int_1^\zeta \frac{h(\eta) - \xi^2 - m^2}{h(\eta)} \mathrm{d}\eta\mathrm{d}\zeta \tag{19}$$

and it follows that, by the definition of h,

$$\phi_0 = L_0$$

$$\phi_1 = K_1 h_0 (R-1) + L_1$$

$$\phi_2 = K_2 h_0 (R-1) + L_2 + \frac{K_1 h_0}{2}(R-1)^2 - \frac{K_1^2 h_0^2}{2}(R-1)^2 - \frac{(\lambda^2 - m^2)}{2}(R-1)^2 \tag{20}$$

for $R \geq 1$, where λ is given in Eq. (4).

Using the same procedure as in the previous section, a three-term outer expansion of the three-term inner expansion of ϕ is derived in the form

$$\phi_{o3}^{i3} = L_0 + \delta[K_1 h_0 (R-1) + L_1] + \delta^2 [K_2 h_0 (R-1) + L_2 - ½(K_1^2 h_0^2 - K_1 h_0 + \lambda^2 - m^2)(R-1)^2] \tag{21}$$

Now, in the cylindrical case, the two linearly independent solutions of Eq. (15) in the uniform core flow can be taken as $J_m(\lambda r)$ and $Y_m(\lambda r)$, the Bessel functions of order m. The inner solution derived previously can be matched with these two functions separately, thus resulting in an asymptotic representation of the general solution of Eq. (15) in the region $r > 1-\delta$ in a manner entirely analogous to the procedure used in Sec. III. However, in this case, it is simpler to recognize at the outset that only the Bessel function of the first kind can be included in the core region if P is to remain bounded at $r=0$. Thus, the solution for P in the outer region is

$$P = J_m(\lambda r)$$

Further, in order to effect a matching of this with the inner solution in the form given by Eq. (17), it is cast into exponential form:

$$P = \exp[\ln J_m(\lambda r)]$$

Hence, the outer expansion of the phase function ϕ of Eq. (17), valid to any order in δ, is

$$\phi^{on} = \ln J_m(\lambda r)$$

The inner expansion of this is calculated by substituting $1-\delta R$ for r and expanding for small δ holding R fixed. Thus,

$$\phi_{i3}^{o3} = \ln J_m(\lambda) - \delta\lambda R \frac{J_m'(\lambda)}{J_m(\lambda)} + \frac{\delta^2 \lambda^2 R^2}{2}\left[\frac{J_m''(\lambda)}{J_m(\lambda)} - \frac{J_m'^2(\lambda)}{J_m^2(\lambda)}\right] \tag{22}$$

According to the matching principle[6] $\phi_{o3}^{i3} = \phi_{i3}^{o3}$, so that by equating Eqs. (21) and (22) one finds

$$L_0 = \ln J_m(\lambda), \quad L_1 = h_0 K_1, \quad L_2 = ½ K_2 h_0$$

$$K_1 = -\frac{\lambda}{h_0}\frac{J_m'(\lambda)}{J_m(\lambda)} = \frac{1}{h_0}\left[m - \frac{\lambda J_{m-1}(\lambda)}{J_m(\lambda)}\right]$$

$$K_2 = \frac{1}{h_0}[K_1 h_0 - K_1^2 h_0^2 - \lambda^2 + m^2] \tag{23}$$

A uniformly valid representation of ϕ can be constructed by the technique of additive composition[6] in the form

$$\phi = \ln J_m(\lambda r) + \sum_{n=0}^{2} \delta^n \phi_n(R) - \phi_{o3}^{i3} + \mathcal{O}(\delta^3) \tag{24}$$

where ϕ_{o3}^{i3} is given by Eq. (21) and ϕ_n are given by Eq. (19). In the following section this expression is used to form the eigenvalue equation for the acoustic wave numbers in a cylindrical duct.

IV. Eigenvalue Equations

To illustrate the use of the solutions derived in the preceding analysis it will be assumed that the duct wall in Fig. 1 is locally reacting so that the boundary condition on P at $y=0$ or $r=1$ is

$$\frac{\mathrm{d}P}{\mathrm{d}n} + i\omega\beta P = 0 \tag{25}$$

Table 1 Wave numbers, ξ: $\beta = 0$, $M_0 = -0.3$, $\omega = 1$, $\delta = 0.01$

	Exact	Eq. (29)		Eq. (30)	
Mode	ξ	ξ	% error	ξ	% error
1^+	1.4255	1.4255	0	1.4255	0
1^-	−0.7697	−0.7697	0	−0.7697	0
2^+	0.3509 − i3.0962	0.3509 − i3.0962	0.0002	0.3509 − i3.0962	0.0008
3^+	0.4223 − i6.4716	0.4224 − i6.4715	0.0014	0.4219 − i6.4716	0.0063
4^+	0.5393 − i9.7733	0.5396 − i9.7731	0.0039	0.5396 − i9.7724	0.0260
5^+	0.7056 − i13.0420	0.7065 − i13.0413	0.0092	0.6965 − i13.0373	0.0785
6^+	0.9307 − i16.2603	0.9328 − i16.2579	0.0195	0.9003 − i16.2479	0.2014
7^+	1.2049 − i19.3545	1.2075 − i19.3475	0.0387	1.1177 − i19.3500	0.4502

in which β is the dimensionless normal admittance of the wall, and n stands for either $-y$ or r. For the two-dimensional case, the condition on $y=1$ will be taken to be $dP/dy=0$, corresponding to symmetric acoustic propagation in a duct of width 2. The analysis for antisymmetric propagation can be done similarly using $P=0$ on $y=1$.

Two-Dimensional Propagation

The general solution for P can be formed as a linear combination of the two solutions in Eq. (14):

$$P=Ce^{\phi^+}+De^{\phi^-}$$

where ϕ^+ and ϕ^- denote the two results included in Eq. (14). The expansions for ϕ will be carried only to second-order accuracy here. After the boundary condition at $y=1$ is applied, P becomes

$$P=2C\exp\left[i\lambda+\delta^2\phi_2(Y)\right]\cos\chi(Y) \tag{26}$$

in which

$$\chi(Y)=\lambda-\delta\lambda\left[1+\int_1^Y\frac{h(\eta)}{h_0}\,d\eta\right] \tag{27}$$

The boundary condition (25) becomes, in terms of Y,

$$\frac{dP}{dY}-i\omega\beta\delta P=0 \quad \text{on} \quad Y=0$$

Differentiation of Eq. (26) and substitution into this condition results in

$$\delta^2\phi_2'(0)\cos\chi(0)-\chi'(0)\sin\chi(0)-i\omega\beta\delta\cos\chi(0)=0 \tag{28}$$

From Eq. (27),

$$\chi'(0)=-\delta\lambda\frac{h(0)}{h_0}=-\delta\frac{\lambda\omega^2}{h_0}$$

and, from Eq. (14),

$$\phi_2'(0)=\frac{\omega^2}{h_0}\int_1^0\left[\lambda^2\left(\frac{h(\eta)}{h_0}-1\right)+\xi^2\left(\frac{h_0}{h(\eta)}-1\right)\right]d\eta$$

Substitution of these expressions into Eq. (28) then yields the desired eigenvalue equation in the form

$$\lambda\tan\left[\lambda(1-\delta)+\lambda\delta\int_0^1\frac{h(\eta)}{h_0}\,d\eta\right]=\frac{i\beta h_0}{\omega}$$
$$+\delta\int_0^1\left[\lambda^2\left(\frac{h(\eta)}{h_0}-1\right)+\xi^2\left(\frac{h_0}{h(\eta)}-1\right)\right]d\eta \tag{29}$$

in which, by Eq. (7),

$$\frac{h(Y)}{h_0}=\left[\frac{\omega-\xi M_0\Phi(Y)}{\omega-\xi M_0}\right]^2$$

$$h_0=\lambda^2+\xi^2=(\omega-\xi M_0)^2$$

Equation (29) is the transcendental relation which yields the axial wave numbers ξ in the two-dimensional duct. Numerical solutions of this equation will be discussed in Sec. V. It is of interest in this context to write the left-hand side of Eq. (29) as

$$\frac{\lambda\tan\lambda(1-\delta)+\lambda\tan\delta\lambda I}{1-\tan\delta\lambda I\,\tan\lambda(1-\delta)}$$

where I denotes the integral in the argument of the tangent in Eq. (29). If the small argument approximation $\tan\delta\lambda I\approx\delta\lambda I$ is used, and if $\mathcal{O}(\delta)$ terms only are retained, then substitution into Eq. (29) yields

$$\lambda\tan\lambda(1-\delta)+\delta\lambda^2 I=\frac{i\beta h_0}{\omega}\left[1-\delta\lambda I\tan\lambda(1-\delta)\right]$$
$$+\delta\lambda^2 I+\delta\int_0^1\left[-\lambda^2+\xi^2\left(\frac{h_0}{h(\eta)}-1\right)\right]d\eta$$

or,

$$\lambda\tan\lambda(1-\delta)=\left[\frac{i\beta h_0}{\omega}-\delta h_0\int_0^1\left[1-\frac{\xi^2}{h(\eta)}\right]d\eta\right]\Bigg/$$
$$\left[1+\delta\frac{i\beta}{\omega}\int_0^1 h(\eta)\,d\eta\right] \tag{30}$$

This is precisely the eigenvalue equation which results if the procedure of Eversman and Beckemeyer[1] is applied to a rectangular duct to derive an equivalent boundary condition for the uniform flow at $y=\delta$. Equation (30) also follows from the analysis of Tester.[3] It will be seen later that Eq. (29) provides more accurate results for the wave numbers ξ than does Eq. (30). One of the reasons for this is the use of the

Table 2 Wave numbers, ξ: $\beta=0$, $M_0=-0.3$, $\omega=1$, $\delta=0.1$

	Exact	Eq. (29)		Eq. (30)	
Mode	ξ	ξ	% error	ξ	% error
1^+	1.3994	1.3995	0.0023	1.3994	0.0058
1^-	−0.7744	−0.7744	0.0004	−0.7744	0.0004
2^+	0.5128 − i2.9983	0.5155 − i2.9929	0.2001	0.5012 − i3.0156	0.6829
3^+	0.8799 − i6.0196	0.8642 − i5.9726	0.8141	0.7715 − i6.1973	3.4210
4^+	0.8273 − i8.9403	0.7554 − i8.9011	0.9119	0.7287 − i9.4279	5.5402
5^+	0.6325 − i12.1981	0.5499 − i12.1919	0.6775	0.5973 − i12.9097	5.8332
6^+	0.5188 − i15.5888	0.4408 − i15.6051	0.5111	0.5096 − i16.5053	5.8764
7^+	0.4562 − i19.0044	0.3572 − i19.0722	0.6310	0.4567 − i20.1388	5.9672

small argument approximation for $\tan\delta\lambda I$ in going from Eq. (29) to Eq. (30). This approximation becomes increasingly inaccurate as $|\xi|$, and, hence, $|\lambda|$, increases.

Cylindrical Propagation

In this case the wall boundary condition is

$$\frac{dP}{dr}+i\omega\beta P=0 \quad \text{on} \quad r=1$$

or

$$\frac{dP}{dR}-i\omega\beta\delta P=0 \quad \text{on} \quad R=0 \tag{31}$$

The uniformly valid solution for P which is bounded at $r=0$ is given by $P=\exp(\phi)$, where ϕ is taken from Eq. (24). Substitution of P into Eq. (31) yields the requirement

$$\phi_1'(0)+\delta\phi_2'(0)=i\omega\beta$$

If Eqs. (19) are used, this becomes

$$K_1+\delta\left[K_2-K_1+K_1^2\int_0^1 h(\eta)\mathrm{d}\eta+\int_0^1\frac{h(\eta)-\xi^2-m^2}{h(\eta)}\mathrm{d}\eta\right]=\frac{i\beta}{\omega}$$

Finally, Eqs. (23) are employed to give

$$\lambda\frac{J_m'(\lambda)}{J_m(\lambda)}+\delta\left[\lambda^2-m^2+\lambda^2\frac{J_m'^2(\lambda)}{J_m^2(\lambda)}\left(1-\int_0^1\frac{h(\eta)}{h_0}\mathrm{d}\eta\right)\right.$$
$$\left.-h_0\int_0^1\frac{h(\eta)-\xi^2-m^2}{h(\eta)}\mathrm{d}\eta\right]=\frac{-i\beta h_0}{\omega} \tag{32}$$

Equation (32) is the relation which determines the wave numbers ξ for the cylindrical duct. It is noted that in both Eqs. (29) and (32) the limiting forms for $\delta\rightarrow 0$ are the standard eigenvalue equations for uniform flow.[5]

V. Numerical Results

To verify the analysis given herein some representative calculations of wave numbers ξ are exhibited in this section. For simplicity only the two-dimensional case [Eq. (29)] is considered here. The shear flow profile was taken as a quarter sine, $\Phi=\sin(\pi Y/2)$, for which the integrals appearing in Eq. (29) can be evaluated explicitly.[1] The transcendental equation was solved to an appropriate level of accuracy using the Newton-Raphson technique with suitable initial guesses for ξ_n.

In each case, Eq. (30) was also solved in the same manner. It is emphasized that the integrals which are required in Eq. (29) are the same as those in Eq. (30), so that no extra labor is involved in solving the new equation. In addition, "exact" numerical values for ξ_n were calculated by the usual shooting process using Runge-Kutta integration and iteration on Eq. (1) and the associated boundary conditions. As is well known, such a direct numerical solution of the boundary value problem can consume substantial computation time. In the present case, especially for the higher-order modes, the time required for the "exact" solution often exceeded that for solution of Eq. (29) by two or more orders of magnitude. The results of the three calculations are displayed here in tabular form in order to make evident the small differences which often occur between them. The first column in Tables 1-5 gives the exact value of ξ, the second contains the approximation of Eq. (29), and the third shows the result predicted by Eq. (30). As a measure of the error in the two approximations, the magnitude of the difference between the exact and the approximate ξ, expressed as a percentage of $|\xi_{\text{exact}}|$, is given in each case.

The first three tables contain the lowest seven wave numbers for a hard-walled case, $\beta=0$. In each, only the ξ_n^+ for positively propagating cutoff modes are shown; the wave number of a negatively propagating cutoff mode ξ_n^- is the complex conjugate of ξ_n^+. Table 1 gives the results for $\omega=1$ and $M_0=-0.3$, with the thickness $\delta=0.01$. It is seen that the error in the approximations increases with mode order, as expected. The maximum error in the solutions of Eq. (30) is less than 0.5%, but the improved approximation yields errors of less than 0.04% in the worst case. For a thicker layer, $\delta=0.1$, Table 2 indicates that the maximum error in the new approximations is less than 1% over the first seven modes, while Eq. (30) yields errors of nearly 6% for modes 4-7. Table 3 corresponds to a higher frequency, $\omega=5$, with $M_0=-0.225$ and $\delta=0.01$. Both approximations are extremely accurate, but the errors in the results from Eq. (30) are consistently at least three times as large as those in the current approximation.

Tables 4 and 5 contain results for a lined duct with $\beta=0.72+i0.42$ at frequency $\omega=1$ and thickness $\delta=0.01$. Again the first seven positively and negatively propagating modes are considered. For $M_0=-0.3$, Table 4 shows errors in the results using Eq. (30) which increase with mode order to values greater than ten times the errors in the solutions of Eq. (29). For a higher Mach number, as in Table 5, it is interesting

Table 3 Wave numbers, ξ: $\beta=0$, $M_0=-0.225$, $\omega=5$, $\delta=0.01$

	Exact	Eq. (29)		Eq. (30)	
Mode	ξ	ξ	% error	ξ	% error
1^+	6.4432	6.4432	0	6.4432	0.0001
1^-	−4.0839	−4.0839	0	−4.0839	0
2^+	5.3369	5.3369	0	5.3369	0
2^-	−2.9815	−2.9815	0	−2.9815	0.0004
3^+	$1.1882-i3.7166$	$1.1882-i3.7166$	0.0002	$1.1882-i3.7168$	0.0041
4^+	$1.2057-i8.1030$	$1.2058-i8.1030$	0.0009	$1.2055-i8.1034$	0.0052
5^+	$1.2299-i11.7568$	$1.2301-i11.7567$	0.0022	$1.2293-i11.7576$	0.0088
6^+	$1.2606-i15.2146$	$1.2612-i15.2143$	0.0044	$1.2590-i15.2161$	0.0144
7^+	$1.2976-i18.5854$	$1.2988-i18.5847$	0.0075	$1.2942-i18.5877$	0.0224

Table 4 Wave numbers, ξ: $\beta = 0.72 + i0.42$, $M_0 = -0.3$, $\omega = 1$, $\delta = 0.01$

	Exact	Eq. (29)		Eq. (30)	
Mode	ξ	ξ	% error	ξ	% error
1^+	$1.3879 - i0.8169$	$1.3879 - i0.8169$	0.0015	$1.3879 - i0.8169$	0.0025
1^-	$-0.9058 + i0.1848$	$-0.9058 + i0.1848$	0.0007	$-0.9058 + i0.1848$	0.0010
2^+	$0.6208 - i3.6612$	$0.6209 - i3.6614$	0.0057	$0.6211 - i3.6609$	0.0114
2^-	$0.4196 + i2.5073$	$0.4196 + i2.5072$	0.0039	$0.4195 + i2.5074$	0.0064
3^+	$0.3579 - i7.1381$	$0.3580 - i7.1386$	0.0084	$0.3598 - i7.1373$	0.0296
3^-	$0.8907 + i5.9013$	$0.8905 + i5.9006$	0.0011	$0.8888 + i5.9025$	0.0377
4^+	$0.2527 - i10.5681$	$0.2530 - i10.5693$	0.0115	$0.2590 - i10.5666$	0.0613
4^-	$1.2128 + i11.8315$	$1.2097 + i11.8307$	0.0270	$1.2107 + i11.8545$	0.1937
5^+	$0.2119 - i13.9716$	$0.2123 - i13.9736$	0.0146	$0.2261 - i13.9688$	0.1038
5^-	$1.0098 + i14.9335$	$1.0065 + i14.9337$	0.0222	$1.0157 + i14.9635$	0.2047
6^+	$0.2092 - i17.3535$	$0.2099 - i17.3565$	0.0175	$0.2352 - i17.3482$	0.1529
6^-	$0.8483 + i18.1796$	$0.8450 + i18.1802$	0.0183	$0.8586 + i18.2158$	0.2072
7^+	$0.2309 - i20.7187$	$0.2318 - i20.7228$	0.0202	$0.2727 - i20.7095$	0.2064
7^-	$0.7393 + i21.4707$	$0.7359 + i21.4715$	0.0159	$0.7526 + i21.5148$	0.2141

Table 5 Wave numbers, ξ: $\beta = 0.72 + i0.42$, $M_0 = -0.5$, $\omega = 1$, $\delta = 0.01$

	Exact	Eq. (29)		Eq. (30)	
Mode	ξ	ξ	% error	ξ	% error
1^+	$1.5284 - i1.0148$	$1.5284 - i1.0148$	0.0016	$1.5284 - i1.0147$	0.0041
1^-	$-0.7723 + i0.1263$	$-0.7723 + i0.1263$	0.0007	$-0.7723 + i0.1263$	0.0003
2^+	$-0.7068 - i4.5428$	$-0.7069 - i4.5431$	0.0059	$0.7079 - i4.5422$	0.0260
2^-	$0.8922 + i2.2221$	$0.8921 + i2.2221$	0.0035	$0.8922 + i2.2223$	0.0099
3^+	$-6.3646 - i3.5510$	$-6.3628 - i3.5529$	0.0353	$-6.3460 - i3.5625$	0.2999
3^-	$1.2386 + i5.6398$	$1.2381 + i5.6397$	0.0071	$1.2389 + i5.6422$	0.0426
4^+	$0.4943 - i8.4867$	$0.4946 - i8.4873$	0.0076	$0.4990 - i8.4839$	0.0644
4^-	$1.1074 + i9.0800$	$1.1067 + i9.0802$	0.0072	$1.1095 + i9.0861$	0.0705
5^+	$0.4717 - i12.3015$	$0.4723 - i12.3023$	0.0084	$0.4821 - i12.2940$	0.1045
5^-	$0.9778 + i12.6821$	$0.9770 + i12.6823$	0.0065	$0.9820 + i12.6928$	0.0903
6^+	$0.4847 - i16.0392$	$0.4856 - i16.0404$	0.0090	$0.5032 - i16.0251$	0.1447
6^-	$0.8978 + i16.3262$	$0.8969 + i16.3266$	0.0060	$0.9044 + i16.3426$	0.1083
7^+	$0.5032 - i19.7389$	$0.5043 - i19.7404$	0.0095	$0.5325 - i19.7165$	0.1869
7^-	$0.8468 + i19.9791$	$0.8457 + i19.9795$	0.0057	$0.8560 + i20.0023$	0.1251

that both the approximations are generally more accurate. Again, however, the errors resulting from the use of Eq. (30) are much larger than those from Eq. (29) (by a factor of 22, for example, for the 7^- mode).

The general trends exhibited by the above sample calculations have been found to be maintained in numerous other computed examples. It is found that for soft-walled cases the new approximation results in an error figure of less than 1% over the first seven modes in every case studied. This is true even for thickness values up to $\delta = 0.1$. For example, for $\delta = 0.1$ and all of the other parameters unchanged in the case illustrated in Table 5, the error figure for the 5^+ mode wave number predicted by Eq. (29) is 0.69%; the approximation of Eq. (30) results in an error of more than 10%.

Acknowledgment

This work was supported by NASA Langley Research Center under NCC1-14.

References

[1]Eversman, W. and Beckemeyer, R., "Transmission of Sound in Ducts with Thin Shear Layers—Convergence to the Uniform Flow Case," *Journal of the Acoustical Society of America*, Vol. 52, No. 1, Pt. 2, 1972, pp. 216-220.

[2]Eversman, W., "Approximation for Thin Boundary Layers in the Sheared Flow Duct, Transmission Problem," *Journal of the Acoustical Society of America*, Vol. 53, No. 5, 1973, pp. 1346-1350.

[3]Tester, B. J., "Some Aspects of Sound Attenuation in Lined Ducts Containing Inviscid Mean Flows with Boundary Layers," *Journal of Sound and Vibration*, Vol. 28, No. 2, 1973, pp. 217-245.

[4]Swinbanks, M. A., "The Sound Field Generated by a Source Distribution in a Long Duct Carrying Sheared Flow," *Journal of Sound and Vibration*, Vol. 40, No. 1, 1975, pp. 51-76.

[5]Nayfeh, A. H., Kaiser, J. E., and Telionis, D. P., "Acoustics of Aircraft Engine-Duct Systems," *AIAA Journal*, Vol. 13, Feb. 1975, pp. 130-153.

[6]Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Annotated Edition, Parabolic Press, Stanford, Calif., 1975.

[7]Nayfeh, A. H., *Introduction to Perturbation Techniques*, John Wiley & Sons, New York, 1980.

From the AIAA Progress in Astronautics and Aeronautics Series . . .

GASDYNAMICS OF DETONATIONS AND EXPLOSIONS—v. 75
and
COMBUSTION IN REACTIVE SYSTEMS—v. 76

Edited by J. Ray Bowen, University of Wisconsin,
N. Manson, Université de Poitiers,
A. K. Oppenheim, University of California,
and R. I. Soloukhin, BSSR Academy of Sciences

The papers in Volumes 75 and 76 of this Series comprise, on a selective basis, the revised and edited manuscripts of the presentations made at the 7th International Colloquium on Gasdynamics of Explosions and Reactive Systems, held in Göttingen, Germany, in August 1979. In the general field of combustion and flames, the phenomena of explosions and detonations involve some of the most complex processes ever to challenge the combustion scientist or gasdynamicist, simply for the reason that *both* gasdynamics and chemical reaction kinetics occur in an interactive manner in a very short time.

It has been only in the past two decades or so that research in the field of explosion phenomena has made substantial progress, largely due to advances in fast-response solid-state instrumentation for diagnostic experimentation and high-capacity electronic digital computers for carrying out complex theoretical studies. As the pace of such explosion research quickened, it became evident to research scientists on a broad international scale that it would be desirable to hold a regular series of international conferences devoted specifically to this aspect of combustion science (which might equally be called a special aspect of fluid-mechanical science). As the series continued to develop over the years, the topics included such special phenomena as liquid- and solid-phase explosions, initiation and ignition, nonequilibrium processes, turbulence effects, propagation of explosive waves, the detailed gasdynamic structure of detonation waves, and so on. These topics, as well as others, are included in the present two volumes. Volume 75, *Gasdynamics of Detonations and Explosions*, covers wall and confinement effects, liquid- and solid-phase phenomena, and cellular structure of detonations; Volume 76, *Combustion in Reactive Systems*, covers nonequilibrium processes, ignition, turbulence, propagation phenomena, and detailed kinetic modeling. The two volumes are recommended to the attention not only of combustion scientists in general but also to those concerned with the evolving interdisciplinary field of reactive gasdynamics.

Volume 75—468 pp., 6×9, illus., $30.00 Mem., $45.00 List
Volume 76—688 pp., 6×9, illus., $30.00 Mem., $45.00 List
Set—$60.00 Mem., $75.00 List

TO ORDER WRITE: Publications Order Dept., AIAA, 1633 Broadway, New York, N.Y. 10019